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Associative rings with metabelian adjoint group

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Abstract

The set of all elements of an associative ring R, not necessarily with a unit element, forms a monoid under the circle operation $r \circ s = r + s + rs$ on R whose group of all invertible elements is called the adjoint group of R and denoted by R°. The ring R is radical if $R = R^{\circ}$. It is proved that a radical ring R is Lie metabelian if and only if its adjoint group R° is metabelian. This yields a positive answer to a question raised by S. Jennings and repeated later by A. Krasil'nikov. Furthermore, for a ring R with unity whose multiplicative group R^* is metabelian, it is shown that R is Lie metabelian, provided that R is generated by R^* and R modulo its Jacobson radical is commutative and artinian. This implies that a local ring is Lie metabelian if and only if its multiplicative group is metabelian.

Keywords: Adjoint group; Associative ring; Lie metabelian ring; Radical ring; Metabelian group

1. Introduction

Let R be an associative ring, not necessarily with a unit element. The set of all elements of R forms a monoid with neutral element $0 \in R$ under the operation $r \circ s = r + s + rs$ for all r and s of R. The group of all invertible elements of this monoid is called the *adjoint group* of R and is denoted by R°. Clearly, if R has a unity 1, then $1 + R^{\circ}$ coincides with

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the multiplicative group R^* of R and the mapping $r \mapsto 1 + r$ with $r \in R$ is an isomorphism from R° onto R^*.

Following Jacobson [4], a ring R is called *radical* if $R = R^\circ$, which means that R coincides with its Jacobson radical. Every associative ring R can also be viewed as a Lie ring under the Lie multiplication $[r, s] = rs - sr$ for all $r, s \in R$. For additive subgroups V and W of R, we denote by VW the *subring of R generated by all products vw* with $v \in V$ and $w \in W$ and by $[V, W]$ the *additive subgroup of R generated by all Lie-commutators* $[v, w]$. The subgroup V is a *Lie ideal* of R if $[V, R] \subseteq V$.

The derived chain of a Lie ring R is defined inductively as $\delta_0(R) = R$ and $\delta_{n+1}(R) = [\delta_n(R), \delta_n(R)]$ for each integer $n \geqslant 0$. The ring R is called *Lie soluble of length $m \geqslant 1$* if $\delta_m(R) = 0$ and $\delta_{m-1}(R) \neq 0$. Lie soluble rings of length at most 2 are called *Lie metabelian*. If $r_1, r_2, \ldots$ are elements of R, the Lie-commutators $[r_1, \ldots, r_{n+1}]$ are defined inductively by $[r_1, \ldots, r_{n+1}] = [[r_1, \ldots, r_n], r_{n+1}]$ for all $n \geqslant 2$. The ring R is *Lie nilpotent of class n* if n is the smallest integer such that $[r_1, \ldots, r_{n+1}] = 0$ for all $r_1, \ldots r_{n+1}$ of R. Recall also that soluble groups and nilpotent group of class n are defined in a corresponding way where the usual group commutator replaces the Lie-commutator. We use brackets to denote Lie-commutators and parentheses to denote group commutators.

It was proved by Jennings [6] that a radical ring R is Lie nilpotent if and only if its adjoint group R° is nilpotent. However, for Lie soluble radical rings, the corresponding result does not hold. For instance, it is well known that the ring R of all (2×2)-matrices over a commutative radical domain of characteristic 2 is radical and satisfies the equality $[\delta_2(R), R] = 0$, whereas the adjoint group of R contains a non-abelian free subgroup and so is non-soluble. Later Zalesskii and Smirnov [13] and independently Sharma and Srivastava [9] have shown that in every Lie soluble ring R the ideal generated by $[\delta_2(R), R]$ is nilpotent.

On the other hand, the authors have proved in [2], Theorem A, that every radical ring R whose adjoint group R° is soluble must be Lie soluble. Furthermore, it was shown by Krasil'nikov [7] and independently by Sharma and Srivastava [10] that the adjoint group of every Lie metabelian ring is metabelian. For rings without 2-torsion this was also proved by Smirnov [11]. Nevertheless the problem *whether a radical ring R is Lie metabelian precisely when the adjoint group R° is metabelian* still remained open, although it has already been raised earlier (see, e.g., [6]). This question was repeated in [7, Question], where an affirmative answer for nil rings was given. The following theorem settles this problem completely.

Theorem A. *Let R be a radical ring. Then the adjoint group R° is metabelian if and only if R is Lie metabelian.*

The ring of all (2×2)-matrices over the Galois field $GF(2)$ whose adjoint group is isomorphic to the symmetric group of degree 3 shows that even a finite ring with metabelian adjoint group need not be Lie metabelian. In fact, every Lie metabelian ring modulo its Jacobson radical must be commutative, so that the following theorem seems to be the best result which can be proved in general.

Recall that a commutative ring is *artinian* if it satisfies the minimal condition for its ideals.

Theorem B. *Let R be a ring with unity whose multiplicative group R^* is metabelian and let J be the Jacobson radical of R. If R is generated by R^* and the factor ring R/J is commutative and artinian, then R is a Lie metabelian ring.*

Note that in Theorem B the condition that R is generated by R^* cannot be omitted. A counterexample is given by the ring of all upper triangular (3×3)-matrices over the field $GF(2)$.

Recall that a ring R with unity is *local* if R modulo its Jacobson radical is a division ring. By a theorem of Hua [3] every division ring with soluble multiplicative group is a field. Hence Theorem B and the above-mentioned result of Krasil'nikov and of Sharma and Srivastava yield as an immediate consequence the following result.

Theorem C. *Let R be a local ring. Then the multiplicative group R^* is metabelian if and only if R is Lie metabelian.*

All notations are mainly standard. The word "ring" means always an associative ring, not necessarily with unity. For this reason in our terminology every ideal is a subring.

2. Lie ideals

Throughout this section $\mathfrak{R}$ denotes the ring obtained by adjoining a formal unity 1 to a ring R when R has no unity, and $\mathfrak{R} = R$ otherwise. Recall that R is an ideal in $\mathfrak{R}$ and every element of $\mathfrak{R}$ can uniquely be written in the form $m + r$ with $m \in \mathbb{Z}$ and $r \in R$.

An ideal I of a ring R will be called *commutative* if $[I, I] = 0$. For a subgroup A of the additive group R^+ of R and subgroups G, H of the multiplicative group $\mathfrak{R}^*$ of the ring $\mathfrak{R}$, we denote by $[A, G]$ the subgroup of R^+ and by (G, H) that of $\mathfrak{R}^*$ generated by all additive commutators $[a, g]$ with $a \in A$ and group commutators (g, h) with $g \in G$ and $h \in H$, respectively.

Lemma 2.1. *Let A be a commutative ideal of a ring R. Then $A[A, R] = 0$ and the ideal A^2 is contained in the centre of R. Moreover, if G is a subgroup of the multiplicative group $\mathfrak{R}^*$ of the ring $\mathfrak{R}$ and A is contained in the Jacobson radical of R (equivalently, $1 + A$ is a subgroup of $\mathfrak{R}^*$), then $(1 + A, G) = 1 + [A, G]$.*

Proof. If $a, c \in A$ and $r \in R$, then $(ar)c = c(ar) = a(cr)$ and therefore $a[c, r] = 0$. Furthermore, $acr = cra = rac$, and so $[ac, r] = 0$.

Next, if $a \in A$ and $g \in G$, then $(1 + a, g) = 1 + (1 + a)^{-1}g^{-1}[a, g] = 1 + g^{-1}[a, g] = 1 + [g^{-1}a, g] \in 1 + [A, G]$. Hence $(1 + A, G) \subseteq 1 + [A, G]$ because $(1 + [a, g])(1 + [b, h]) = 1 + [a, g] + [b, h]$ for any $b \in A$ and $h \in G$. Conversely, if $b = ga$, then $b \in A$ and $[a, g] = [g^{-1}b, g] = g^{-1}[b, g] = (1 + b)^{-1}g^{-1}[b, g]$, so that $1 + [a, g] = (1 + b, g)$. Thus $1 + [A, G] \subseteq (1 + A, G)$, as desired. □

An ideal I of R is said to be *Lie nilpotent* if $[I, {}_nI] = 0$ for some positive integer n. It was already proved by Jennings [5] that in every Lie-nilpotent ring R the ideal generated by $[R, R, R]$ is nilpotent. The following generalization of this can also be found in Streb [12].

Lemma 2.2. *Let I be a Lie nilpotent ideal of a ring R. Then the ideal of R generated by $[I, R, R]$ is nilpotent.*

Proof. Clearly it suffices to prove that the ideal $[I, R, R]\mathfrak{R}$ is nilpotent modulo a nilpotent ideal of R, so that we can assume that such an ideal is zero. In particular, we may suppose that $[I, I, I]\mathfrak{R} = 0$ because the ideal $[I, I, I]\mathfrak{R}$ is nilpotent by the above-mentioned result of Jennings [5]. Taking now $a, b.c, d \in I$ and $r, s \in R$ and substituting the elements $u = [r, a, b]$, $x = s$, $y = c$ and $z = d$ in the identity

$$u[x, y, z] = [ux, y, z] - [u, y][x, z] - [u, z][x, y] - [u, y, z]x,$$

we obtain

$$\begin{aligned}[r, a, b][s, c, d] = \big[[r, a, b]s, c, d\big] - \big[[r, a, b], c\big][s, d] - \big[[r, a, b], d\big][s, c] \\ + \big[[r, a, b], c, d\big]s.\end{aligned}$$

Since all summands of the right side of this equality belong to $[I, I, I]\mathfrak{R} = 0$, it follows that $[r, a, b][s, c, d] = 0$ and therefore $[R, I, I]^2 = 0$. But then $([R, I, I]\mathfrak{R})^2 = 0$ and we may again assume that $[R, I, I]\mathfrak{R} = 0$. As $[r, a, s]b = [r, a, sb] - s[r, a, b] = 0$, this implies $[R, I, R]I = 0$ and hence $[I, R, R]^2 = 0$ because $[I, R, R] = [R, I, R] \subseteq I$. Thus $([I, R, R]\mathfrak{R})^2 = 0$. □

The following assertion is due to Sharma and Srivastava [9].

Lemma 2.3. *Let R be a ring and S a subring of R. If S is a Lie ideal of R, then $[S, S, S]^2 \subseteq [[S, S], [S, S]]\mathfrak{R}$. In particular, if R is Lie metabelian, then $[R, R, R]^2 = 0$.*

Proof. Indeed, if $a, b, c, r, s, t \in S$, then $[a, b, c][r, s, t] = [[a, b, c], [r, s]]t + [[a, b]t, c, [r, s]] - [a, b][[c, t], [r, s]] - [[a, b], [r, s]][c, t] \in [[S, S], [S, S]]\mathfrak{R}$. □

Recall that the *Levitzki radical* of a ring R is the unique maximal locally nilpotent ideal of R.

Lemma 2.4. *Let R be a ring whose adjoint group is metabelian, J the Jacobson and L the Levitzki radical of R. If the factor ring R/J is commutative, then the ideal of R generated by $[R, R, R]$ is nilpotent and the factor ring R/L is likewise commutative.*

Proof. Since the adjoint group of J is metabelian, the ideal J is commutative modulo L by [2, Theorem B], and L is a Lie metabelian ring by the result of Krasil'nikov [7] mentioned in the introduction. Therefore $[L, L, L]^2 \subseteq [[L, L], [L, L]]\mathfrak{R} = 0$ by Lemma 2.3

and so the ideal $[L, L, L]\mathfrak{R}$ is nilpotent. Passing to the factor ring $R/[L, L, L]\mathfrak{R}$, we may assume that $[L, L, L] = 0$. Then L is Lie nilpotent and hence the ideal $I = [L, R, R]\mathfrak{R}$ is nilpotent by Lemma 2.2. As $[J, J] \subseteq L$, this implies that $[J, J, J, J] \subseteq I$ and thus J/I is Lie nilpotent. A repeated application of Lemma 2.2 yields that the ideal of R generated by $[J, R, R]$ is nilpotent modulo I and so itself is nilpotent. Passing again to the factor ring $R/[J, R, R]\mathfrak{R}$, we may now suppose that $[J, R, R] = 0$. Since $[R, R] \subseteq J$, it follows that $[R, R, R, R] = 0$ and hence R is Lie nilpotent. Therefore the ideal $[R, R, R]\mathfrak{R}$ is nilpotent by Lemma 2.2 and the factor ring R/L is commutative by [1, Main Theorem]. □

3. Rings of generalized quotients

Throughout this section R will be a ring with unity 1. Let S be a subring of R and $U = R^* \cap (1 + S)$. Put $U^{-1} = \{u^{-1} \mid u \in U\}$. We say that R is the *ring of generalized quotients* of S if the union $S \cup U^{-1}$ generates R as a ring. It is easy to see that R is the ring of generalized quotients of every subring of R containing S and, for every ideal I of R, the factor ring R/I is the ring of generalized quotients of its subring $(S + I)/I$.

For an additive subgroup V of R, we denote by VU^{-1} and $U^{-1}VU^{-1}$ the subsets $\{vu^{-1} \mid v \in V,\ u \in U\}$ and $\{t^{-1}vu^{-1} \mid v \in V,\ t, u \in U\}$ of R, respectively. Similarly we define the subset $U^{-1}V$. We say that R is the *ring of quotients* of S if $R = U^{-1}S = SU^{-1}$. Clearly in this case $1 \in S$.

Lemma 3.1. *Suppose that the ring R contains a Lie nilpotent subring S and let $U = R^* \cap (1+S)$. Then the set SU^{-1} forms a Lie nilpotent subring of R such that $SU^{-1} = U^{-1}S$ and the subring SU^{-1} is Lie nilpotent if S is finitely generated. In particular, if S contains* 1, *then R is the ring of generalized quotients of S if and only if R is the ring of quotients of S.*

Proof. Clearly the set SU^{-1} is a subring of R if for every two elements $r \in S$ and $u \in U$ there exist elements $s \in S$ and $v \in U$ such that $u^{-1}r = sv^{-1}$.

Put $r_0 = r$ and $r_{m+1} = [r_m, u]$ for each $m \geqslant 0$. Then $r_m \in S$ and $u^{-1}r_m u = r_m + u^{-1}r_{m+1}$ for every m. Since the subring S is Lie nilpotent, there exists a positive integer n such that $r_n = 0$. Hence $u^{-1}r_{n-1} = r_{n-1}u^{-1}$ and this implies that

$$u^{-1}r = \bigl(ru^n + r_1u^{n-1} + \cdots + r_{n-1}u + r_n\bigr)u^{-(n+1)}.$$

Putting $s = ru^n + r_1u^{n-1} + \cdots + r_{n-1}u + r_n$ and $v = u^{n+1}$, we have $s \in S$ and $u^{-1}r = sv^{-1}$, as claimed. By symmetry, the set $U^{-1}S$ is also a subring of R and so $SU^{-1} = U^{-1}S$.

We show now that the subring SU^{-1} is Lie nilpotent if S is finitely generated. Obviously it suffices to consider the case when the subring S contains 1. Then it follows from [1, Lemmas 2.3 and 4.1], that the Jacobson radical J of S is nilpotent and the factor ring S/J is commutative. Therefore $JU^{-1} = U^{-1}J$ is an ideal of SU^{-1} and the factor ring $(SU^{-1})/(JU^{-1})$ is also commutative. Furthermore, if $J^n = 0$ for some positive integer n and Z is the centre of S, then the intersection $I = J^{n-1} \cap Z$ is an ideal of S by [1, Lemma 2.4]. Hence $IU^{-1} = U^{-1}I$ is a central ideal of SU^{-1} because $[IU^{-1}, SU^{-1}] = I[U^{-1}, SU^{-1}] \subseteq I(JU^{-1}) = 0$. Passing to the factor ring

$(SU^{-1})/(IU^{-1})$ and arguing by induction on n, we conclude that the subring SU^{-1} is Lie nilpotent. $\square$

Lemma 3.2. *Let R be the ring of generalized quotients of a finitely generated subring of R. If the multiplicative group of R is metabelian and R is commutative modulo its Jacobson radical, then the Levitzki radical of R is nilpotent.*

Proof. Let L be the Levitzki radical of R and let S be a finitely generated subring of R containing 1 such that R is the ring of generalized quotients of S. Since the ideal I of R generated by $[R, R, R]$ is nilpotent by Lemma 2.4, the passage to the factor ring R/I allows us to assume that R is Lie nilpotent. Then $R = U^{-1}S = SU^{-1}$ with $U = R^* \cap S$ by Lemma 3.1 and so $L = U^{-1}(L \cap S) = (L \cap S)U^{-1}$. As the ideal $L \cap S$ of S is nilpotent by [1, Lemma 4.2], this implies that L is also nilpotent. $\square$

Lemma 3.3. *Let R be the ring of generalized quotients of a subring S of R with $U = R^* \cap (1 + S)$ and let I be an ideal of R. Then the following statements hold.*

(1) *If $R = U^{-1}SU^{-1}$, then $I = U^{-1}(I \cap S)U^{-1}$.*
(2) *If the factor ring R/I is Lie nilpotent and K is an ideal of S such that $IK = KI = 0$, then the set $U^{-1}KU^{-1}$ is an ideal of R.*

Proof. Indeed, since $R = U^{-1}SU^{-1}$, for each $a \in I$, there exist elements $s \in S$ and $t, u \in U$ such that $a = t^{-1}su^{-1}$. Therefore $s = tau \in I \cap S$ and hence $a \in U^{-1}(I \cap S)U^{-1}$, so that statement (1) is proved.

Next, if the factor ring R/I is Lie nilpotent and S_1 is the subring of R generated by S and 1, then $R = I + U^{-1}S_1 = I + S_1U^{-1}$ by Lemma 3.1. Therefore $RK = U^{-1}K$ and hence $RKR = (U^{-1}K)(I + S_1U^{-1}) = U^{-1}KU^{-1}$. This proves statement (2). $\square$

Lemma 3.4. *Let R be the ring of generalized quotients of a subring S of R and $U = R^* \cap (1 + S)$. If there exists an ideal I of R such that $I^2 = 0$ and the factor ring R/I is Lie nilpotent, then the set $U^{-1}SU^{-1}$ is an ideal of R. Moreover, if S contains 1, then $R = U^{-1}SU^{-1}$.*

Proof. Note first that the set $J = U^{-1}(I \cap S)U^{-1}$ forms an ideal of R by Lemma 3.3. Next, $I \cap S = J \cap S$ because $I \cap S \subseteq J \subseteq I$. Put $\overline{R} = R/J$ and use bars for homomorphic images in $\overline{R}$. Then $\overline{S}$ is isomorphic to the factor ring $S/(I \cap S)$ and so is a Lie nilpotent ring. Therefore $\overline{U}^{-1}\overline{S} = \overline{S}\,\overline{U}^{-1}$ is a subring of $\overline{R}$ by Lemma 3.1. But then $J + U^{-1}S = J + SU^{-1}$ is a subring and so an ideal of R because R is generated by S and U^{-1}. As $U^{-1}SU^{-1} = J + U^{-1}S$, the lemma is proved. $\square$

Recall that an abelian group M is an (R, R)-*bimodule* if M is simultaneously a left R-module and a right R-module with $(ra)s = r(as)$ for all $a \in M$ and $r, s \in R$.

Lemma 3.5. *Let the ring R be commutative and M a finitely generated (R, R)-bimodule. If R is the ring of quotients of a finitely generated subring of R, then M is a noetherian*

(R, R)-bimodule. Furthermore, if R is finitely generated and M is a simple (R, R)-bimodule, then M is finite.

Proof. Suppose first that R itself is a finitely generated ring with $n \geqslant 1$ generators. Then R is a homomorphic image of the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of all polynomials with integer coefficients in n commuting indeterminates under a ring homomorphism α. The bimodule M can be made into a module over the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ in $2n$ indeterminates as follows.

If $a \in M$ and $f = gh$ is a monomial of $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ written as the product of two monomials g and h in indeterminates $x_1, \ldots, x_n$ and $y_1 \ldots, y_n$, respectively, then the element $af \in M$ is defined by $af = g^\alpha a h^\alpha$ and this definition is expanded linearly to any polynomial f of $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. It is easily verified that in this case M, regarded as a $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$-module, is a finitely generated module, a noetherian module or a simple module, respectively, if M has the same property as an (R, R)-bimodule. Since every finitely generated module over a polynomial ring is noetherian by Hilbert's Basis Theorem and every simple module over this ring is finite by Hilbert's Nullstellensatz, in the case under consideration the lemma is proved.

In the general case, let S be a finitely generated subring of R containing 1 whose ring of quotients is R. If $M = Ra_1R + \cdots + Ra_mR$ for elements $a_1, \ldots, a_m$ of M, then the (S, S)-bimodule $N = Sa_1S + \cdots + Sa_mS$ is noetherian by what was proved above. Therefore, for every (R, R)-subbimodule L of M, the intersection $L \cap N$ is a finitely generated (S, S)-bimodule, so that $L \cap N = Sb_1S + \cdots + Sb_lS$ for some elements $b_1, \ldots, b_l$ of $L \cap N$. Furthermore, for every $a \in L$, there are elements $r_1, u_1, \ldots, r_m, u_m$ of R such that $a = r_1a_1u_1 + \cdots + r_ma_mu_m$. As R is a ring of quotients of S, there exist two elements $s, t \in R^* \cap (1 + S)$ such that all products $sr_1, u_1t, \ldots, sr_m, u_mt$ belong to S and hence the element $sat = sr_1a_1u_1t + \cdots + sr_ma_mu_mt$ belongs to $L \cap N$. Thus $a \in s^{-1}(L \cap N)t^{-1} \subseteq Rb_1R + \cdots + Rb_lR \subseteq L$ and therefore $L = Rb_1R + \cdots + Rb_lR$ is a finitely generated (R, R)-bimodule, so that M is noetherian. □

4. Weakly noetherian rings

We say that a ring R is *weakly noetherian* if R satisfies the maximum condition for its ideals. The ideal of R generated by $[R, R]$ is called the *commutator ideal* of R. Obviously this is the smallest ideal of R modulo which R is commutative.

Throughout this section we assume that R is a ring with unity 1.

Lemma 4.1. *Let R be the ring of generalized quotients of a finitely generated subring of R. If R is Lie metabelian, then R is weakly noetherian.*

Proof. Assume first that R itself is finitely generated as a ring and let K be the subring of R generated by $[R, R]$. Then K is commutative and contains the ideal I of R generated by $[R, R, R]$ because $[r, s, t]u = [[r, s]u, t] + [r, s][t, u] \in K$ for all $r, s, t, u \in R$. In particular, $[K, I] = 0$ and, furthermore, $I^2 = 0$ by Lemma 2.3. Since the factor ring R/I is Lie nilpotent and finitely generated, it is noetherian by [1, Lemma 4.2], and so there exists a finite subset X of I such that $I = RXR$. We show that the ideal I, regarded as an (R, R)-bimodule, is also noetherian.

If J is the commutator ideal of R and m a positive integer, then $I \subseteq J$ and the ideal J^m, regarded as a one-sided ideal of R, is finitely generated modulo I. Therefore there exist a finite subset Y of J^m such that $J^m + I = RY + I$. As $J^m = R[R, R]^m = [R, R]^m R$, we may even assume that $Y \subseteq K$. Then $J^m I = (J^m + I)I = (RY)I = (RY)(RXR) = RXYR = IJ^m$ because $[I, Y] = 0$. Hence the ideal $J^m I$ of R and so the (R, R)-bimodule $J^m I/J^{m+1}I$ is finitely generated for each $m \geqslant 0$. Moreover, since $J^m I/J^{m+1}I$ can be regarded as an $(R/J, R/J)$-bimodule and R/J is a finitely generated commutative ring, each of these bimodules is noetherian by Lemma 3.5. But then I is also a noetherian (R, R)-bimodule because the ideal J is nilpotent modulo I by a result of Jennings [5] and so there exists a positive integer n such that $J^n = 0$. Thus the ring R is weakly noetherian, as claimed.

Consider now the general case and let S be a finitely generated subring of R containing 1 such that R is the ring of generalized quotients of S. If $U = R^* \cap S$, then $R = U^{-1}SU^{-1}$ by Lemma 3.4. Hence, if N is an ideal of R, then $N = U^{-1}(N \cap S)U^{-1}$ by Lemma 3.3 and the intersection $N \cap S$ is finitely generated as an ideal of S by what was proved above. Therefore the ideal N is also finitely generated and thus the ring R is weakly noetherian. □

Lemma 4.2. *Let R be a ring whose multiplicative group R^* is metabelian. Suppose that R is the ring of generalized quotients of a finitely generated subring of R and let the commutator ideal I of R be contained in the Jacobson radical of R. If R is generated by R^* and n is a positive integer, then the ideal I^n is finitely generated and the factor ring R/I^n is weakly noetherian.*

Proof. Note first that the factor ring R/I is noetherian because it is in a commutative ring of quotients of one of its finitely generated subrings. Next, for each positive integer m, the ideal I^m/I^{m+1} of the factor ring R/I^{m+1} can be viewed as an $(R/I, R/I)$-bimodule. It is also clear that the ring R/I^{n+1} is weakly noetherian for a certain $n \geqslant 1$ if and only if the $(R/I, R/I)$-bimodule I^m/I^{m+1} is noetherian for each $1 \leqslant m \leqslant n$. Since every finitely generated $(R/I, R/I)$-bimodule is noetherian by Lemma 3.5, it suffices to show that the $(R/I, R/I)$-bimodules I^m/I^{m+1} are finitely generated. Clearly this holds if I^m is finitely generated as an ideal of R for each $1 \leqslant m \leqslant n$. Since every factor ring R/I^{m+1} satisfies the hypothesis of the lemma, we may argue by induction on n and assume that I^m has this property for every $m \leqslant n - 1$, so that R/I^n is a weakly noetherian ring. We have to prove that I^n is also finitely generated as an ideal of R. Obviously without loss of generality we may henceforth suppose that $I^{n+1} = 0$.

As R is generated by R^*, there exists a finitely generated subgroup G of R^* such that R is the ring of generalized quotients of the subring generated by G. Therefore R is commutative modulo the ideal $R((G, G) - 1)R$ generated by the set $(G, G) - 1$. Considering that for every two elements $g, h \in G$ the equality $(g, h) - 1 = g^{-1}h^{-1}[g, h]$ holds, we conclude that the ideal $R((G, G) - 1)R$ coincides with I. Hence there exists a finite subset $X \subseteq (G, G) - 1$ such that $I = RXR$. In particular, the case $n = 1$ is completed.

Let $n \geqslant 2$ and Y be a finite subset of I^{n-1} such that $I^{n-1} = RYR$. Obviously the centralizer $C_R(X) = \{r \in R \mid [r, X] = 0\}$ of X in R contains I^n. Furthermore,

$(1 + I^{n-1}, R^*) = 1 + [I^{n-1}, R^*]$ modulo I^n by Lemma 2.1. Therefore $[I^{n-1}, R] = [I^{n-1}, R^*] \subseteq C_R(X)$. Thus, putting $K = [I^{n-1}, R] + I^n$, we have $[K, X] = 0$. Note that K is an ideal of R. Indeed, if $a \in I^{n-1}$ and $r, s, t \in R$, then $r[a, s]t = ra[s, t] + [rat, s] + [s, r]at \in K$. Hence I^{n-1} is a central ideal of R modulo K and so $I^{n-1} = RY + K$. Moreover, as the factor ring R/I^n is weakly noetherian, there exists a finite subset $Z \subseteq K$ such that $K = RZR + I^n$. Therefore $IK = (RXR)(RZR + J^n) = RXRZR = RZXR = RXZR = KI$ because $I^n I = II^n = 0$ and $[X, RZ] = 0$. Thus IK is finitely generated as an ideal of R and IK is contained in I^n.

Put $\overline{R} = R/IK$ and let bars denote homomorphic images modulo IK. Then $\overline{I}\,\overline{K} = \overline{K}\,\overline{I} = 0$ and so $\overline{K}$ can be viewed as a finitely generated $(R/I, R/I)$-bimodule. Therefore this bimodule is noetherian by Lemma 3.5 and hence its $(R/I, R/I)$-subbimodule $\overline{I}^n$ is finitely generated. But then $\overline{I}^n$ is finitely generated as an ideal of $\overline{R}$ and therefore I^n has the same property as an ideal of R, as claimed. □

We say that an ideal I of a ring R is *co-finite* if the factor ring R/I is finite.

Lemma 4.3. *Let R be the ring of generalized quotients of a finitely generated subring of R. If R contains a co-finite nilpotent ideal, then R is finite.*

Proof. Assume first that the ring R is finitely generated. Then every co-finite ideal of R is finitely generated as a subring of R by [8]. Therefore every co-finite nilpotent ideal of R is finitely generated as an additive subgroup of R, so that the additive group of R is also finitely generated. On the other hand, this group has finite exponent because $m \cdot 1 \in I$ for some positive integer m and so $n \cdot 1 = 0$ for some power n of m. Thus R is finite.

Consider now the general case. By what has been proved above, every finitely generated subring of R is finite, so that R is the ring of generalized quotients of a finite subring S of R. But then the set $U = R^* \cap (1 + S)$ is finite and so R as the ring generated by the union $S \cup U^{-1}$ is also finite. □

Recall that a ring R is *subdirectly irreducible* if the intersection of all non-zero ideals of R is a non-zero ideal of R usually called the *monolith* of R.

Lemma 4.4. *Let R be the ring of generalized quotients of a finitely generated subring of R. Assume that R is weakly noetherian and its commutator ideal J is nilpotent. If R is subdirectly irreducible and the monolith of R is finite, then R is finite.*

Proof. Assume the contrary and let R be a counterexample whose monolith M is finite. Clearly the two-sided annihilator $A(M) = \{r \in R \mid rA = 0 = Ar\}$ of M in R is a co-finite ideal of R. We show first that $J \neq 0$.

Indeed, otherwise R is commutative and therefore R is the ring of quotients of a finitely generated subring S containing 1. Thus, if $U = R^* \cap S$, then $R = U^{-1}S$ and so $M = U^{-1}(M \cap S)$. Since S is residually finite as a ring by [1, Corollary 4.4], there exists a co-finite ideal N of S such that $(M \cap S) \cap N = 0$. But then $U^{-1}N$ is an ideal of R and $M \cap U^{-1}N = 0$. Hence $N = 0$ and so the subring S is finite. Clearly then R is also finite, contrary to the assumption.

Put now $J^0 = R$ and let n be the smallest non-negative integer such that the ideal J^{n+1} is finite. Clearly the set $P = \{r \in R \mid J^n r J = 0\}$ forms an ideal of R. Moreover, as $J^n RJ \subseteq J^{n+1}$, the ideal $J^n RJ$ is finite and so P is a co-finite ideal of R. By symmetry, the ideal $Q = \{r \in R \mid JrJ^n = 0\}$ of R is also co-finite, as well as the two-sided annihilator $A(J^{n+1})$ of J^{n+1} in R. Thus the intersection $T = A(M) \cap A(I) \cap P \cap Q$ is a co-finite ideal of R such that $JTJ^n = J^n TJ = 0$. Denote by A the two-sided annihilator of J^n in T and show that the factor ring T/A is nilpotent. By symmetry, it suffices to prove that T modulo the right annihilator $\mathrm{Ann}_T(J^n)$ is nilpotent. Obviously this will hold if the factor ring $T/\mathrm{Ann}_T(J^n)$ is a nil ring.

Suppose this is not the case, so that there exists an element $r \in T$ such that $\mathrm{Ann}_T(J^n) \cap \{r^m \mid m \geqslant 1\} = \emptyset$. If $s \in T$ and $t \in R$, then $[sr,t] = s[r,t] + [s,t]r \in TJ + JT$ and so $(J^n sr)t = (J^n t)sr + (J^n)[sr,t] \subseteq J^n sr$ because $J^n(TJ + JT) \subseteq J^n TJ + J^{n+1}T = 0$. Therefore $J^n r^m$ is an ideal of R for every integer $m \geqslant 2$ and hence the intersection $\bigcap_{m=2}^{\infty} J^n r^m$ contains M. Thus, if a is a non-zero element of M, then for each $m \geqslant 2$ there exist elements $a_m \in J^n$ such that $a = a_m r^m$. As R is weakly noetherian, the ideal of R generated by the set $\{a_m \mid m \geqslant 2\}$ is finitely generated. This means that there exist an integer $l \geqslant 2$ and elements $\{r_i, s_i \mid 2 \leqslant i \leqslant l\}$ of $\mathfrak{R}$ such that $a_{l+1} = \sum_{i=2}^{l} r_i a_i s_i$. But then $a = a_{l+1} r^{l+1} = \sum_{i=2}^{l} r_i a_i s_i r^{l+1} = \sum_{i=2}^{l} (r_i a_i [s_i, r^{l+1}] + r_i a_i r^{l+1} s_i) = 0$ because $r_i a_i [s_i, r^{l+1}] = 0$ and $a_i r^{l+1} = a r^{l+1-i} = 0$ for every $2 \leqslant i \leqslant l$, contrary to the choice of the element a.

Thus T/A is a nilpotent ring. Furthermore, T/A is a co-finite ideal of the factor ring R/A which is the ring of generalized quotients of a finitely generated subring of R/A. Therefore the additive group of R/A and hence of the factor ring R/A is finitely generated by Lemma 4.3. Thus there exists a finitely generated subgroup K of the additive group of R such that $R = A + K$. Since R is a weakly noetherian ring, there exists a finite subset X of J^n such that $J^n = RXR$. Therefore $J^n = (A + K)X(A + K) = KXK$ because $AJ^n = J^n A = 0$. This means that J^n is finitely generated as an additive subgroup of R and hence $\bigcap_{m=1}^{\infty} mJ^n = 0$. As mJ^n is an ideal of R for every integer m, it follows that $mJ^n = 0$ for some m and thus the ideal J^n must be finite, contrary to the choice of n. This final contradiction completes the proof. □

Recall that R is *residually finite* as a ring if the intersection of all co-finite ideals of R is zero and R is *residually nilpotent* if $\bigcap_{n=1}^{\infty} R^n = 0$.

Proposition 4.5. *Let R be a finitely generated ring whose commutator ideal J is residually nilpotent. If R is weakly noetherian, then R is residually finite as a ring.*

Proof. Obviously we may restrict ourselves to the case when the ideal J is nilpotent. Furthermore, we may also assume that R is a subdirectly irreducible ring, and we have to show that in this case R is finite. Let M be the monolith of R. Since M is a minimal ideal of R, it is contained in the two-sided annihilator of J and so $JM = MJ = 0$. Therefore M can be viewed as an $(R/J, R/J)$-bimodule and hence M is finite by Lemma 3.5. But then R must be finite by Lemma 4.4. □

5. Finite rings

For every subset A of a ring R, we denote by $C_R(A)$ the centralizer of A in R, i.e., $C_R(A) = \{r \in R \mid ra = ar$ for every $a \in A\}$.

Lemma 5.1. *Let R be a ring with unity and A a subgroup of order n of the multiplicative group R^*. If n is invertible in R and I is an ideal of R, then $I = [I, A] + C_I(A)$ and $[I, A] \cap C_I(A) = 0$. In particular, $[I, A] = [I, A, A]$.*

Proof. Clearly the mapping $\alpha : I \to I$ given by $r^\alpha = \sum_{a\in A} a^{-1}ra$ for every $r \in I$ is an endomorphism of the additive group of I whose image is contained in $C_I(A)$. Furthermore, for every $b \in A$, it follows that

$$[r, b]^\alpha = \sum_{a\in A} a^{-1}[r, b]a = \left[\sum_{a\in A} a^{-1}ra, b\right] = 0.$$

Thus, if $[r, b] \in C_I(A)$, then

$$0 = [r, b]^\alpha = \sum_{a\in A}[r, b] = n[r, b]$$

and so $[r, b] = 0$. Since

$$r^\alpha - nr = \sum_{a\in A}\bigl(a^{-1}ra - r\bigr) = \sum_{a\in A}\bigl[a^{-1}r, a\bigr] \in [I, A],$$

this implies that $nr \in [I, A] + C_I(A)$ and from $nI = I$ it follows that $I = [A, I] + C_I(A)$. In particular, $[I, A] = [I, A, A] + C_{[A,I]}(A) = [I, A, A]$ because $[I, A]$ is an ideal in the subring of R generated by A and $[I, A]$. □

Recall that an ideal I of R has the *Artin–Rees property* if for each finitely generated R-module M and its submodule N there exists a positive integer n such that $MI^n \cap N \subseteq NI$.

Lemma 5.2. *Let R be a subdirectly irreducible finite ring with unity, I a nilpotent ideal of R and G a metabelian subgroup of the multiplicative group R^* of R such that R is generated by G as a ring. If the factor ring R/I is commutative and $1 + I \subseteq G$, then there exist an ideal K of R contained in I and a Lie nilpotent local subring S of R such that $R = K + S$, $K^2 = K \cap S = 0$ and $K = [K, S]$. Moreover, if $K \neq 0$, then $I \cap S$ is contained in the centre of S and so $[S, S, S] = 0$. In particular, the ring R is local.*

Proof. Since R has only one minimal ideal M which lies in every non-zero ideal of R, the additive group of R is a p-group for some prime number p. Furthermore, M is contained in I and also in the two-sided annihilator of the Jacobson radical J of R, so that $MJ = JM = 0$.

Since the factor ring R/J is a direct sum of fields of characteristic p, the multiplicative group $(R/J)^*$ of R/J is abelian and its order is not divisible by p. As $(R/J)^*$ is isomorphic to the factor group $R^*/(1+J)$, the subgroup $(1+J)G$ is decomposed in a semidirect product $(1+J) \rtimes A$ of its normal p-subgroup $1+J$ and a non-trivial abelian p'-subgroup $A \subseteq G$ by the well-known Schur–Zassenhaus theorem. Furthermore, G is metabelian, so that the last term U of the lower central series of G is an abelian p-subgroup. This implies that G splits over U and all complements to U in G are conjugate by a theorem of Schenkman [8]. Hence there exists a nilpotent subgroup B of G containing A such that $G = U \rtimes B$ and $(U, B) = U$. Clearly $B \subseteq C_G(A)$ and $U \subseteq 1+I$, so that $G = (1+I)C_G(A)$ and $1+I = U \rtimes (B \cap (1+I))$. Moreover, because the subgroup $C_U(A)B$ is nilpotent, (A, U) is a normal subgroup in G and $U = (A, U) \times C_U(A)$, we derive that $U = (A, U)$ and $B = C_G(A)$, so that $1+I = U \rtimes C_{1+I}(A)$.

On the other hand, putting $K = [I, A]$, we have that $I = K + C_I(A)$, $K \cap C_I(A) = 0$ and $[K, A] = K$ by Lemma 5.1. This means in particular that K and U have the same order. Let N be the subring of I generated by the set $U - 1$. Then N is commutative and invariant under the action of A by conjugation. Thus $N = (K \cap N) \oplus C_N(A)$ and obviously $1 + N = U \times C_{1+N}(A)$, so that the orders of $K \cap N$ and U must also be equal. Therefore $K \cap N = K$ and hence $K \subseteq N$. We show that $K^2 = 0$ and K is in fact an ideal of R.

Let C be the subring of R generated by A and $S = C_R(A)$. Then $S = J + C$ and C is contained in the centre of S, so that S is a Lie nilpotent ring. Furthermore, $R = I + S$ because $G = (1+I)C_G(A)$. Therefore $R = K + S$ and $K \cap S = 0$, so that $K = [K, C] = [R, C]$. This implies that $KS \subseteq K$ because $[r, c]s = [rs, c]$ for every $c \in C$, $r \in R$ and $s \in S$. Hence $KR = K + K^2 \subseteq N$ and, by symmetry, $RK = K + K^2 \subseteq N$. Thus $RKR \subseteq N$ is a commutative ideal of R and therefore $RKR[RKR, R] = 0$ by Lemma 2.1. As $K = [K, C] \subseteq [RKR, R]$, it follows that $K^2 = 0$ and so $RKR = K$.

Assume now that K is a non-zero ideal of R, so that $M \subseteq K$. If T is the two-sided annihilator of K in S, then $RTR = (K+S)T(K+S) = STS \subseteq T$ and so T is an ideal of R. As $M \cap T \subseteq K \cap T = 0$, this implies $T = 0$. Therefore the centralizer $C_S(K)$ of K in S contains no non-zero commutative ideals of S. Indeed, if L is such an ideal, then $K + L$ is a commutative ideal of R, so that $(K+L)[K+L, R] = 0$ by Lemma 2.1 and hence $(K+L)K = K(K+L) = 0$, contrary to the above considerations.

Next let L be a commutative ideal of S such that $L \subseteq I \cap S$ and L is maximal with this property. Clearly $B = C_G(A) \subseteq S$, so that $(1+L, B) = 1 + [L, B]$ and also $(1+K, G) = 1 + [K, G]$ by Lemma 2.1. Furthermore, $K = [K, G]$ because $K = [K, A] \subseteq [K, G] \subseteq K$. Since the group G is metabelian, it follows that $(1+L, B, 1+K) = ((1+L, B), (1+K, G)) = 1$ and so $[L, B] \subseteq C_S(K)$. Obviously $[L, B]$ is a C-module because $C[L, B]C = [CLC, B] = [L, B]$. Therefore the subring Q generated by $[L, B]$ is also a C-module. It is clear that $Q \subseteq C_S(K)$, so that $K + Q$ is a commutative ideal of the subring $K + Q + C$. As above, this means that $0 = (K+Q)[K+Q, K+Q+C]$ and so $(K+Q)K = 0$ because $K = [K, C] \subseteq [K+Q, K+Q+C]$. Thus $Q = 0$ and so $[L, B] = 0$. Therefore L is contained in the centre of S and hence $L = I \cap S$ by the choice of L. Since $[S, S] \subseteq I \cap S$, this implies that $[S, S, S] = 0$.

We show finally that R and so S are local rings. Clearly it suffices to prove that the right annihilator $\mathrm{Ann}_R(M)$ is nilpotent because the factor ring $R/\mathrm{Ann}_R(M)$ is a field. But if R is Lie nilpotent, then every ideal of R has the Artin–Rees property by [1, Lemma 4.2]. Therefore there exists a positive integer n such that $\mathrm{Ann}_R(M)^n \cap M \subseteq M \cdot \mathrm{Ann}_R(M) = 0$ and hence $\mathrm{Ann}_R(M)^n = 0$. In the other case $R = K + S$ and so

$\mathrm{Ann}_R(M) = K + \mathrm{Ann}_S(M)$. As S is Lie nilpotent and K is an S-module, we have $(K \cdot \mathrm{Ann}_S(M)^n) \cap M \subseteq M \cdot \mathrm{Ann}_S(M) = 0$ and so $K \cdot \mathrm{Ann}_S(M)^n = 0$. By symmetry, also $\mathrm{Ann}_S(M)^n \cdot K = 0$, so that $\mathrm{Ann}_S(M)^n = 0$ because the two-sided annihilator of K in S is zero. Thus $\mathrm{Ann}_R(M) = J$ and so the ring R is local. □

Corollary 5.3. *Let R be a finite ring with unity whose multiplicative group R^* is metabelian and let J be the Jacobson radical of R. If the factor ring R/J is commutative and R^* generates R as a ring, then the ring R is Lie metabelian.*

Proof. Clearly it suffices to consider the case when R is a subdirectly irreducible ring. Then $R = K + S$ for an ideal K and a Lie nilpotent local subring S of R such that $K^2 = K \cap S = 0$ and $K = [K, S]$ by Lemma 5.2. If $K = 0$, then $[R, R] = [J, J]$ and so $[[R, R], [R, R]] = [[J, J], [J, J]] = 0$ by the result of Krasil'nikov mentioned in the introduction. Otherwise, putting $I = J$ in Lemma 5.2, we obtain that the intersection $J \cap S$ and so the subring S itself are commutative. Therefore $[R, R] = K$ and so R is Lie metabelian. □

6. Rings with large Lie nilpotent homomorphic images

Throughout this section R will denote a ring with unity 1. For each positive integer n, we denote by $\gamma_n(R)$ the ideal of R generated by $[\underbrace{R, \dots, R}_{n}]$.

Lemma 6.1. *Let R be the ring of generalized quotients of a finitely generated subring of R. Suppose that R is a weakly noetherian ring containing a nilpotent ideal I such that the factor ring R/I is Lie nilpotent. Then the following statements hold.*

(1) *There exists a finitely generated subring S of R containing 1 such that $R = U^{-1}SU^{-1}$ with $U = R^* \cap S$.*
(2) *If R is subdirectly irreducible and M is the monolith of R, then either $[M, R] = 0$ or there exists a finitely generated subring S of R such that $M \cap (\bigcap_{n=1}^{\infty} \gamma_n(S)) \neq 0$.*

Proof. To prove (1) take the smallest positive integer n such that $I^n = 0$ and proceed by induction on n. By Lemma 3.1, there exists a finitely generated subring S of R containing 1 such that $R = I + SU^{-1} = I + U^{-1}S$, so that the case $n = 1$ is clear. By induction hypothesis it suffices to show that $I^{n-1} \subseteq U^{-1}SU^{-1}$. Since the ideal I^{n-1} is finitely generated, we may without loss of generality assume that S contains a full set of generators of I^{n-1} as an ideal of R. Then $I^{n-1} = R(I^{n-1} \cap S)R = (I + U^{-1}S)(I^{n-1} \cap S)(I + SU^{-1}) = U^{-1}S(I^{n-1} \cap S)SU^{-1} \subseteq U^{-1}SU^{-1}$ and thus $R = U^{-1}SU^{-1}$.

Assume now that the hypothesis of statement (2) holds and that $[M, R] \neq 0$. Then $I \neq 0$ and so $M \subseteq I$. Furthermore, if L is the Levitzki radical of R, then $ML = LM = 0$. As $I \subseteq L$, the factor ring R/L is commutative by [1, Main Theorem], and thus $M[R, R] = [R, R]M = 0$. Therefore, for every $a \in M$ and all $r, s, t \in R$, it follows that $a[s, t] = [s, t]a = 0$ and hence $s[a, r]t = [sat, r]$. This means that the commutator subgroup $[M,r]=\{[a,r] \mid a\in M\}$ is an ideal of R and so, for some $r\in R$, the equality $[M,r]=M$ holds. In particular, $a=[b,r]$ for some $b\in M$. It is also clear that $M=RaR$ for each non-zero $a\in M$, so that there exist finitely many elements $s_1,t_1,\dots,s_n,t_n$ of R such that $b=\sum_{i=1}^n s_i a t_i$. Therefore $a=[b,r]=[\sum_{i=1}^n s_i a t_i, r]=\sum_{i=1}^n s_i[a,r]t_i$ and hence, for every positive integer m, it follows that

$$a=\sum_{i_1,\dots,i_m} s_{i_1}\dots s_{i_m}[a,\underbrace{r,\dots,r}_{m}]t_{i_1}\dots t_{i_m}.$$

Thus, if S is the subring of R generated by the elements $a,r,s_1,t_1,\dots,s_n,t_n$, then $a\in\gamma_m(S)$ for every $m\geqslant 1$ and so $a\in M\cap(\bigcap_{m=1}^{\infty}\gamma_m(S))$. □

Lemma 6.2. *Let R be the ring of generalized quotients of a finitely generated subring of R and let the multiplicative group R^* of R be metabelian. If R is commutative modulo its Jacobson radical, every proper homomorphic image of R is Lie metabelian and if R is generated by R^*, then $\gamma_5(R)^2=0$.*

Proof. Assume the contrary and let R be a counterexample with $I=\gamma_3(R)$. Then the ideal I is nilpotent by Lemma 2.4 and $M=I^2\neq 0$. Therefore M is the unique minimal ideal of R by Lemma 2.3 and so R is a subdirectly irreducible ring whose monolith M satisfies the condition $MI=IM=0$. Furthermore, R is a weakly noetherian ring by Lemma 4.1. Hence there exists a finitely generated subring S of R such that $R=U^{-1}SU^{-1}$ with $U=R^*\cap(1+S)$ and either $[M,R]=0$ or $M\cap(\bigcap_{n=1}^{\infty}\gamma_n(S))\neq 0$ by Lemma 6.1.

Suppose first that $[M,R]=0$ and show that then the ideal $[I,R]R$ is commutative. Note that in this case M is in fact a minimal one-sided ideal of R and so is annihilated by the Jacobson radical of R. As R is commutative modulo this radical, it follows that $M[R,R]=[R,R]M=0$ and so $[I[R,R],I[R,R]]\subseteq I^2[R,R]=M[R,R]=0$. Furthermore, $[I[R,R],[I,R]]=0$. Indeed, if $a,b\in I$ and $r,s,t\in R$, then $[a[r,s],[b,t]]=a[[r,s],[b,t]]+[a,[b,t]][r,s]=0$ because the elements $[[r,s],[b,t]]$ and $[a,[b,t]]$ are contained in M. Therefore it remains to show that $[[I,R],[I,R]]=0$ because $[a,r]s=[as,r]-a[s,r]$ for each $a\in I$ and any $r,s\in R$.

Since R is generated by R^*, we have $[I,R]=[I,R^*]$. Next, $1+I\subseteq R^*$ and hence $(1+I,R^*)=1+[I,R^*]$ modulo M by Lemma 2.1. As the commutator subgroup $(1+I,R^*)$ of R^* is abelian, this implies that $[[I,R^*],[I,R^*]]=0$, as desired.

Thus the ideal $[I,R]R$ is commutative and therefore $([I,R]R)[[I,R]R,R]=0$ by Lemma 2.1. But then $[I,R,R]^2=0$ and hence $\gamma_5(R)^2=0$, contrary to the assumption.

Now let $[M,R]\neq 0$ and let T be the subring of R containing S such that T is generated by a finitely generated subgroup of R^*. Clearly if $P=\bigcap_{n=1}^{\infty}\gamma_n(T)$, then $M\cap P\neq 0$ and $R=U^{-1}TU^{-1}$. Since T is residually finite by Proposition 4.5, every ideal K of T modulo which T is subdirectly indecomposable is co-finite. Moreover, either $\gamma_n(T)\subseteq K$ for some $n\geqslant 1$ or $\gamma_3(T)^2\subseteq K$ and $\gamma_3(T)=\gamma_4(T)+K$ by Lemma 5.2. Therefore, if Q is the intersection of all co-finite ideals of T containing $\gamma_3(T)^2$, then $P\cap Q=0$. We show that $M\cap Q=0$.

Indeed, both $U^{-1}(M\cap P)U^{-1}$ and $U^{-1}(M\cap Q)U^{-1}$ are ideals of R by Lemma 3.3 and it is easily verified that their intersection coincides with $U^{-1}(M\cap P\cap Q)U^{-1}=0$. As

$M \cap P \neq 0$ and every non-zero ideal of R contains M, this implies $U^{-1}(M \cap Q)U^{-1} = 0$ and so $M \cap Q = 0$.

Finally, $\gamma_3(T)^2 \subseteq \gamma_3(R)^2 = M$ and $\gamma_3(T)^2 \subseteq Q$, so that $\gamma_3(T)^2 \subseteq M \cap Q = 0$ and thus $I^2 = \gamma_3(R)^2 = 0$. But then $M = I^2 = 0$ and this contradiction completes the proof. $\square$

7. Radical rings

Let R be a radical ring, regarded as an ideal of the ring $\mathfrak{R} = R + \mathbb{Z}$ obtained by adjoining a formal unity 1 to R. For a subset S of R, we define the *radical join* of S in R to be the smallest radical subring of R containing S. It is clear that if this subring coincides with R, then $\mathfrak{R}$ is the ring of generalized quotients of S.

The following sums up what the foregoing results imply for radical rings.

Lemma 7.1. *Let R be a radical ring whose adjoint group R° is metabelian and let G be a finitely generated subgroup of R°. If R is the radical join of G, then the following statements hold.*

(1) *The ring $\mathfrak{R}$ and the subring of $\mathfrak{R}$ generated by $1 + G$ are weakly noetherian.*
(2) *Every finitely generated subring of $\mathfrak{R}$ is residually finite as a ring.*
(3) *If every proper homomorphic image of $\mathfrak{R}$ is Lie metabelian, S is a subring of R and I an ideal of S, then the adjoint group of the factor ring S/I is metabelian.*

Proof. Let P be the subring of $\mathfrak{R}$ generated by $1 + G$. To prove (1) note first that the multiplicative group P^* of P is metabelian because $P^* = (1 + (P \cap R)^\circ) \times \mathbb{Z}^*$ and $(P \cap R)^\circ$ is a subgroup of R°. It is also obvious that P^* generates P as a ring. Furthermore, the Levitzki radical L of P is nilpotent by Lemma 3.2 and the factor ring P/L is commutative by Lemma 2.4. Therefore P is weakly noetherian by Lemma 4.2. As $\mathfrak{R}$ is the ring of generalized quotients of P, similar arguments show that R is also a weakly noetherian ring.

Next, the subring P is residually finite by Lemma 4.5. Since every finitely generated subring of $\mathfrak{R}$ is contained in the subring generated by a subgroup of $1 + R$ with finitely many generators, statement (2) is proved.

Finally, if every proper homomorphic image of $\mathfrak{R}$ is Lie metabelian, then $\mathfrak{R}$ satisfies the hypothesis of Lemma 6.2 and so $\gamma_5(\mathfrak{R})^2 = 0$. Thus, for the radical join T of a subring S of R, the subring $\mathfrak{T} = T + \mathbb{Z}$ of $\mathfrak{R}$ satisfies the hypothesis of Lemma 3.4. This implies that $\mathfrak{T} = (1 + S)^{-1}(S + \mathbb{Z})(1 + S)^{-1}$ and therefore $T = (1 + S)^{-1}S(1 + S)^{-1}$. In particular, if I is an ideal of S and Q is the radical join of I in R, then $Q = (1 + I)^{-1}I(1 + I)^{-1}$. We show that $Q + S$ is a subring of R and Q is an ideal of $Q + S$ such that $Q \cap S = I$.

Indeed, let $s \in S$ and $r = (1 + a)^{-1}b(1 + c)^{-1} \in Q$ with $a, b, c \in I$. Then $s(1 + a)^{-1} = s - (sa)(1 + a)^{-1}$ and so $sr = (s - sa(1 + a)^{-1})b(1 + c)^{-1} = (sb)(1 + c)^{-1} - (sa)(1 + a)^{-1}b(1 + c)^{-1} \in Q$ because $sa, sb \in I$. Therefore $SQ \subseteq Q$ and, by symmetry, $QS \subseteq Q$. Furthermore, if $r \in S$, then $(1 + a)r(1 + c)b = b$ and so $r = b - ar - rc - arc \in I$.

Now, the element $s + Q$ belongs to the adjoint group of the factor ring $(Q + S)/Q$ if $(1 + t)(1 + s) = (1 + s)(1 + t) \in 1 + Q$ for some $t \in S$. As $1 + Q$ is a subgroup of $\mathfrak{R}^*$, this implies that the element $1 + s$ is invertible in $1 + Q + S$ and so s belongs to the adjoint group of the subring $Q + S$. Since this group as a subgroup of R° is metabelian, so is the adjoint group of the factor ring $(Q + S)/Q$ which is isomorphic to S/I. This proves statement (3). □

Proof of Theorem A. Clearly it suffices to prove that every radical ring which is the radical join of a finitely generated subgroup of R° is Lie metabelian.

Suppose the contrary and let R be such a counterexample. Then $\mathfrak{R}$ is a weakly noetherian ring by Lemma 7.1(1) and therefore $\mathfrak{R}$ contains an ideal N such that every proper homomorphic image of $\mathfrak{R}/N$ is Lie metabelian but R/N is not. Obviously without loss of generality we may assume that $N = 0$.

Let S be an arbitrary finitely generated subring of R. Since S is residually finite by Lemma 7.1(2) and the adjoint group of every finite homomorphic image of S is metabelian by Lemma 7.1(3), the subring S and so R is Lie metabelian by Lemma 5.3, contrary to the assumption. This contradiction completes the proof. □

8. Proof of Theorem B

Before proving the theorem we recall two well-known facts concerning commutative rings which are freely used.

First, every element of a commutative artinian ring is either invertible or a zero divisor. This implies, in particular, that the multiplicative group of every homomorphic image of a ring R satisfying the hypothesis of Theorem B is metabelian.

Second, if R is a commutative integral domain with unity and I is a proper ideal of R, then the Krull intersection theorem says that $\bigcap_{n=0}^{\infty} I^n = 0$.

Furthermore, we need the following lemma.

Lemma 8.1. *Let R be a ring with* 1 *whose multiplicative group R^* is metabelian, J the Jacobson radical and S a finitely generated subring of R, $U = R^* \cap (1 + S)$ and T the subring of R generated by the union $S \cup U^{-1}$. If R is generated by R^* and the factor ring R/J is commutative, then $J \cap T$ is contained in the Jacobson radical of T.*

Proof. Let L be the Levitzki radical of R and I the Jacobson radical of T. Then the factor ring R/L is commutative by Lemma 2.4 and $L \cap T \subseteq I$. It is also clear that T is the ring of generalized quotients of S and T is generated by its multiplicative group T^*. Therefore $L \cap T$ is a nilpotent ideal of T by Lemma 3.2 and hence T is a weakly noetherian ring by Lemma 4.2. Thus there exists a finitely generated subring P of T containing S and 1 such that $T = V^{-1}PV^{-1}$ with $V = R^* \cap P$ by Lemma 6.1(1), and so $J \cap T = V^{-1}(J \cap P)V^{-1}$ by Lemma 3.3. Now, if $r \in J \cap T$, then $r = u^{-1}tv^{-1}$ for some $u, v \in V$ and $t \in J \cap P$, so that $u(1 + r)v = uv + t \in R^* \cap P = V$. Therefore $1 + r = u^{-1}wv^{-1}$ with $w = uv + t$ and hence $(1 + r)^{-1} = vw^{-1}u \in T \cap (1 + J) = 1 + (J \cap T)$. This means that $J \cap T \subseteq I$, as claimed. □

Proof of Theorem B. Suppose the contrary and let R be a counterexample with Jacobson radical J. As R is generated by its multiplicative group R^*, there exists a subring S of R such that S is not Lie metabelian and S is generated by a subgroup of R^* with finitely many generators. Moreover, since the factor ring R/J is a direct sum of finitely many fields, we may even assume that S modulo J is a direct sum of its ideals $S_1, \ldots, S_n$ such that each S_i modulo J is an integral domain. Put $U = R^* \cap S$ and let T be the subring of R generated by the union $S \cup U^{-1}$. Then T is the ring of generalized quotients of S, the factor ring $T/(J \cap T)$ is artinian and $J \cap T$ is contained in the Jacobson radical of T by Lemma 8.1. Therefore the subring T is artinian modulo its Jacobson radical and hence we may assume that $R = T$. Then the Levitzki radical L of R is nilpotent by Lemma 3.2, the factor ring R/L is commutative by Lemma 2.4 and so R is a weakly noetherian ring by Lemma 4.2.

Let M be the intersection of all ideals of R modulo which R is Lie metabelian. Then M is a non-zero ideal of R and thus there exists an ideal N of R which is properly contained in M and which is maximal with this property. Passing to the factor ring R/N, we may assume that $N = 0$ which means that every proper homomorphic image of R is Lie metabelian. Therefore $\gamma_5(R)^2 = 0$ by Lemma 6.2. Furthermore, the finitely generated subring S of R is residually finite by Proposition 4.5. Hence there exists a co-finite ideal I of S such that the factor ring S/I is not Lie metabelian. Moreover, since S modulo J is the direct sum of its ideals $S_1, \ldots, S_m$, we may even choose the co-finite ideal I such that I modulo J is the direct sum of the ideals $I \cap S_1, \ldots, I \cap S_m$.

Put $V = R^* \cap (1 + I)$ and let P be the subring of R generated by the union $I \cup V^{-1}$. Since the factor ring $P/(P \cap \gamma_5(R))$ is Lie nilpotent and $\gamma_5(R)^2 = 0$, the set $Q = V^{-1}IV^{-1}$ is an ideal of P by Lemma 3.4. We show that $Q + S$ is a subring of R and Q is an ideal of $Q + S$ such that $Q \cap S = I$.

Indeed, if $a \in I$, $u, v \in V$ and $s \in S$, then $vs = s + b$ for some $b \in I$ and so $a(v^{-1}s) = a(s - v^{-1}b) = as - av^{-1}b \in Q$. Thus $(u^{-1}av^{-1})s \in Q$ and, by symmetry, $s(u^{-1}av^{-1}) \in Q$, so that $QS \subseteq S$ and $SQ \subseteq Q$. Furthermore, $Q \cap S = I$ because from $s = u^{-1}av^{-1}$ it follows that $a = usv = s + c$ for some $c \in I$ and therefore $s \in I$.

We show next that the multiplicative group of the factor ring $(Q + S)/Q$ is a homomorphic image of the multiplicative group of the subring $Q + S$. Clearly it suffices to prove that every element $r \in S$ which is invertible modulo Q is invertible in $Q + S$. In other words, we have to show that if $rs = sr \in 1 + I$ for some $s \in S$, then rs is invertible in $1 + Q$.

Assume the contrary and note that in this case the element rs cannot be invertible in R. Indeed, otherwise $rst = 1$ for some $t \in R^*$ and so $t^{-1} = 1 + a$ for some $a \in I$. Hence $t^{-1} \in V$ and thus $t \in V^{-1} \subseteq 1 + Q$, contrary to the assumption. Thus $rs \notin R^*$. Since the factor ring R/J is artinian, this means that rs is a zero divisor modulo J and so there exists $t \in R \setminus J$ such that $(rs)t \in J$. As $R = J + SU^{-1}$ by Lemma 3.1, it follows that $t = x + yu^{-1}$ for some $x \in J$, $y \in S$ and $u \in U$. Furthermore, $rs = 1 + a$ for some $a \in I$. Therefore $(rs)t = (1+a)t = x + yu^{-1} + ax + ayu^{-1} = (x + ax) + (y + ay)u^{-1}$ and hence $(rs)y = y + ay \in J$ because $(rs)t \in J$ and $x + ax \in J$. Thus without loss of generality we may suppose that $t = y \in S$. Using bars for homomorphic images in R/J, we conclude now that $\bar{t} = -\bar{a}\bar{t} \in \bar{I}$. Obviously this implies that $\bar{t} = (-\bar{a})^n \bar{t} \in \bar{I}^n$ for each positive integer n, so that $\bar{t} \in \bigcap_{n=1}^{\infty} \bar{I}^n$. However $\bigcap_{n=1}^{\infty} \bar{I}^n = 0$ because $\bar{I} = \overline{I \cap S_1} \oplus \cdots \oplus \overline{I \cap S_m}$

by the choice of I and $\bigcap_{n=1}^{\infty}(\overline{I \cap S_i})^n = 0$ for each i by the Krull intersection theorem. Hence $\bar{t} = 0$ and so $t \in J$, contrary to the choice of t.

Thus the element r is invertible in $Q + S$ and therefore the multiplicative group of $(Q + S)/Q$ is a homomorphic image of that of $Q + S$. Since $Q \cap S = I$ and so $(Q + S)/Q$ is isomorphic to the factor ring S/I, this means that the multiplicative group of S/I is metabelian. But then S/I is a Lie metabelian ring by Corollary 5.3, contrary to the choice of the ideal I. This contradiction completes the proof. □

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